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RECIPROCAL SKEW MORPHISMS OF CYCLIC GROUPS

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Abstract. A skew morphism of the cyclic additive group $\mathbb{Z}_n$ is a bijection φ on $\mathbb{Z}_n$ for which there exists an integer-valued function $\pi\colon \mathbb{Z}_n \to \mathbb{Z}$ such that $\varphi(0) = 0$ and $\varphi(x + y) = \varphi(x) + \varphi^{\pi(x)}(y)$ for all $x, y \in \mathbb{Z}_n$. A pair of skew morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ are reciprocal if (a) the orders of φ and $\widetilde{\varphi}$ divide m and n, respectively, and (b) the associated power functions π and $\widetilde{\pi}$ are determined by $\pi(x) = \widetilde{\varphi}^x(1)$ and $\widetilde{\pi}(y) = \varphi^y(1)$. Reciprocal pairs of skew morphisms of cyclic groups are in one-to-one correspondence with isomorphism classes of regular dessins with complete bipartite underlying graphs. In this paper we determine all reciprocal pairs of skew morphisms of the cyclic groups provided that one of them is a group automorphism.

1. Introduction

A *dessin* is an embedding $i\colon \Gamma \hookrightarrow \mathcal{C}$ of a connected 2-coloured bipartite graph Γ into an orientable closed surface $\mathcal{C}$ such that each component of $\mathcal{C} \setminus i(\Gamma)$ is homeomorphic to the open disc. An automorphism of a dessin is a colouring-preserving automorphism of the underlying graph which extends to an orientation-preserving self-homeomorphism of the supporting surface. The automorphism group of a dessin acts semi-regularly on the edges. In the case where this action is transitive, and hence regular, the dessin is called *regular* as well.

A skew morphism of a finite group A is a permutation φ on A fixing the identity element of A and for which there exists an integer-valued function $\pi\colon A \to \mathbb{Z}$ such that $\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$ for all $g, h \in A$. In general, the function π is not uniquely determined by φ. However, if φ has order k, then the function π can be viewed as a function $\pi\colon A \to \mathbb{Z}_k$. In this case the function π is unique, and it will be called the *power function of* φ. In particular, if $\pi(\varphi(g)) = \pi(g)$, for all $g \in A$, then the skew morphism φ will be called *smooth*.

Let φ and $\widetilde{\varphi}$ be skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, and let π and $\widetilde{\pi}$ be the power functions of φ and $\widetilde{\varphi}$, respectively. The pair of skew morphisms φ and $\widetilde{\varphi}$ are *reciprocal* if they satisfy the following two numerical

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conditions: (a) the orders of φ and $\widetilde{\varphi}$ divide m and n, respectively, and (b) $\pi(x) = \widetilde{\varphi}^x(1)$ and $\widetilde{\pi}(y) = \varphi^y(1)$ for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$. It was proved in [**8**, Theorem 5] that the isomorphism classes of regular dessins with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with the reciprocal pairs of skew morphisms φ and $\widetilde{\varphi}$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Thus, to classify complete regular dessins it suffices to determine reciprocal pairs of skew morphisms of cyclic groups.

In this paper we prove that, in a reciprocal pair of skew morphisms, if one of the skew morphisms is an automorphism, then the other skew morphism must be smooth. Employing the theory of smooth skew morphisms all such reciprocal pairs are completely determined, see Theorem 14.

2. Preliminaries

In this section we summarize some preliminary results concerning skew morphisms, which will be used throughout the paper.

Let φ be a skew morphism of a finite group A, and let π be the power function of φ, and let k be the order of φ. A subgroup N of A will be called *φ-invariant* if $\varphi(N) = N$. In this case the restriction of φ to N is a skew morphism of N. It is well known that the set $\operatorname{Fix}\varphi = \{x \in A \mid \varphi(x) = x\}$ of fixed points of φ is a φ-invariant subgroup of A. Another important subgroup is $\operatorname{Ker}\varphi = \{x \in A \mid \pi(x) = 1\}$, called the *kernel* of φ [**11**]. Note that, for any $g, h \in A$, $\pi(g) = \pi(h)$ if and only if $gh^{-1} \in \operatorname{Ker}\varphi$, so the power function takes exactly $|A \colon \operatorname{Ker}\varphi|$ distinct values. The number $|A \colon \operatorname{Ker}\varphi|$ is called the *skew type* of φ. It follows that φ is an automorphism of A if and only if it has skew type 1. A skew morphism which is not an automorphism will be termed *proper*.

Recently, Zhang proved that the set

$$\operatorname{Core}\varphi = \bigcap_{i=1}^{k} \varphi^i(\operatorname{Ker}\varphi)$$

is a φ-invariant normal subgroup of A [**21**], which will be called the *core* of φ. This is the largest φ-invariant subgroup of A contained in the kernel of φ. A skew morphism φ of A is *kernel-preserving* if $\operatorname{Ker}\varphi$ is a φ-invariant subgroup. It follows that a skew morphism φ is kernel-preserving if and only if $\operatorname{Ker}\varphi = \operatorname{Core}\varphi$, in which case the restriction of φ to $\operatorname{Ker}\varphi$ is an automorphism of $\operatorname{Ker}\varphi$.

Lemma 1 ([**11**]). *Let φ be a skew morphism of a finite group A, let π be the power function of φ, and let k be the order of φ. Then, for any $x, y \in A$,*

$$\varphi^\ell(xy) = \varphi^\ell(x)\varphi^{\sigma(x,\ell)}(y) \qquad \text{and} \qquad \pi(xy) \equiv \sigma(y, \pi(x)) \pmod{k},$$

where ℓ is an arbitrary positive integer and $\sigma(x, \ell) = \sum_{i=1}^{\ell} \pi(\varphi^{i-1}(x))$.

Lemma 2 ([**20**])**.** *Let φ be a skew morphism of a finite group $A=\langle g_1, g_2, \ldots, g_r\rangle$. Then*

$$|\varphi| = \mathrm{lcm}(|O_{g_1}|, |O_{g_1}|, \ldots, |O_{g_r}|),$$

where O_{g_i} denotes the orbit of φ containing g_i, $i = 1, 2, \ldots, r$. Moreover, φ and π are completely determined by the action of φ and the values of π on the generating orbits $O_{g_1}, O_{g_2}, \ldots, O_{g_r}$.

Lemma 3 ([**23**])**.** *Let φ be a skew morphism of a finite group A, and let N be a φ-invariant normal subgroup of A. Define $\bar{\varphi}$ as $\bar{\varphi}(\bar{g}) = \overline{\varphi(g)}$ for any $\bar{g} \in A/N$, then $\bar{\varphi}$ is a skew morphism of A/N with power function $\bar{\pi}$ determined by $\bar{\pi}(\bar{g}) \equiv \pi(g) \pmod{m}$ where $m = |\bar{\varphi}|$.*

Lemma 4 ([**4**, Lemma 5.1])**.** *Let φ be a skew morphism of an abelian group A, then φ is kernel-preserving, and the restriction of φ to $\mathrm{Ker}\,\varphi$ is an automorphism of $\mathrm{Ker}\,\varphi$.*

The following correspondence between skew morphisms and cyclic complementary factorisations of finite groups is fundamental.

Lemma 5 ([**3**])**.** *If $G=AC$ is a factorisation of a finite group G with $A\cap C=1$ and $C = \langle c\rangle$, then c induces a skew morphism φ of the subgroup A via the commuting rule $cx = \varphi(x)c^{\pi(x)}$; in particular $|\varphi| = |C\colon C_G|$ where $C_G = \cap_{g\in G} C^g$.*

Conversely, if φ is a skew morphism of a finite group A, then $G = L_A\langle\varphi\rangle$ is a transitive permutation group on A with $L_A \cap \langle\varphi\rangle = 1$ and $\langle\varphi\rangle$ core-free in G, where L_A is the left regular representation of A.

A characterisation of the subgroup $\mathrm{Ker}\,\varphi$ can be found in [**3**], the following result is an extension.

Proposition 6. *Let $G=AC$ be a factorisation of a finite group G with $A\cap C=1$ and $C = \langle c\rangle$, and let φ be the skew morphism induced by c via the commuting rule $cx = \varphi(x)c^{\pi(x)}$ for all $x \in A$. Then $A \cap c^{-1}Ac \le \mathrm{Ker}\,\varphi$ and $A_G \le \mathrm{Core}\,\varphi$. Moreover, if $C_G = 1$, then $A \cap c^{-1}Ac = \mathrm{Ker}\,\varphi$ and $A_G = \mathrm{Core}\,\varphi$.*

Proof. For any $x \in A \cap c^{-1}Ac$, there is an element $y \in A$ such that $x = c^{-1}yc$, so $cx = yc$. Since $cx = \varphi(x)c^{\pi(x)}$, we get $yc = \varphi(x)c^{\pi(x)}$, and hence $\pi(x) \equiv 1 \pmod{|c|}$. Since $|\varphi| = |C\colon C_G|$, $|\varphi|$ divides $|c|$. Thus $\pi(x) \equiv 1 \pmod{|\varphi|}$, and hence $x \in \mathrm{Ker}\,\varphi$. This proves $A \cap c^{-1}Ac \le \mathrm{Ker}\,\varphi$.

Moreover, since $G = AC$ and C is cyclic, we have

$$A_G = \bigcap_{g\in G} A^g = \bigcap_{k=1}^{|c|} A^{c^k}.$$

Thus, for any $x \in A_G$ and for every integer k, $1 \le k \le |c|$, there exists $y_k \in A$ such that $c^k x = y_k c^k$. On the other hand, by induction we deduce from the identity $cx = \varphi(x)\pi^{\pi(x)}$ that $c^k x = \varphi^k(x)c^{\sigma(x,k)}$ where

$$\sigma(x, k) = \sum_{i=1}^{k} \pi(\varphi^{i-1}(x)).$$

Thus $y_k c^k = \varphi^k(x)c^{\sigma(x,k)}$. If $k = 1$ then $y_1 c = \varphi(x)c^{\pi(x)}$, so $\pi(x) \equiv 1 \pmod{|c|}$. Assume $\pi(\varphi^i(x)) \equiv 1 \pmod{|c|}$ for all $i \le k$, then from $y^{k+1}c^{k+1} = \varphi^{k+1}c^{\sigma(x,k+1)}$ we deduce that

$$k+1 \equiv \sigma(x,k+1) = \sum_{i=1}^{k+1} \pi(\varphi^{i-1}(x)) = k + \pi(\varphi^k(x)) \pmod{|c|},$$

so $\pi(\varphi^k(x)) \equiv 1 \pmod{|c|}$. By induction $\pi(\varphi^k(x)) \equiv 1 \pmod{|c|}$ for all nonnegative integer k. Since $|\varphi|$ divides $|c|$, $\pi(\varphi^k(x)) \equiv 1 \pmod{|\varphi|}$. Therefore $x \in \operatorname{Core}\varphi$. This proves $A_G \le \operatorname{Core}\varphi$.

Now assume in addition $C_G = 1$. Then by Lemma 5, $|\varphi| = |c|$. For any $x \in \operatorname{Ker}\varphi$, we have $\pi(x) \equiv 1 \pmod{|c|}$, so $cx = \varphi(x)c^{\pi(x)} = \varphi(x)c$. Thus $x = c^{-1}\varphi(x)c \in A \cap c^{-1}Ac$. Therefore $\operatorname{Ker}\varphi = A \cap c^{-1}Ac$. Similarly, for any $x \in \operatorname{Core}\varphi$, $\pi(\varphi^k(x)) \equiv 1 \pmod{|c|}$ for all nonnegative integers k. Then $c^k x = \varphi^k(x)c^{\sigma(x,k)} = \varphi^k(x)c^k$, so $x = c^{-k}\varphi^k(x)c^k \in A_G$. Therefore, $\operatorname{Core}\varphi = A_G$, as required. □

Since $\operatorname{Core}\varphi$ is a φ-invariant normal subgroup of A, by Lemma 3, φ induces a skew morphism $\bar{\varphi}$ of the quotient group $A/\operatorname{Core}\varphi$. Define

$$\operatorname{Smooth}\varphi = \{x \in A \mid \bar{\varphi}(\bar{x}) = \bar{x}\}.$$

It was proved in [**20**] that $\operatorname{Smooth}\varphi$ is a φ-invariant subgroup of A containing $\operatorname{Fix}\varphi$. In the extremal case where $\operatorname{Smooth}\varphi = A$, the skew morphism φ is called a *smooth* skew morphism.

Lemma 7 ([**20**]). *A skew morphism φ of a finite group A is smooth if and only if $\pi(\varphi(g)) = \pi(g)$ for all $g \in A$.*

The most important properties of smooth skew morphisms are summarised as follows.

Lemma 8 ([**20**]). *Let φ be a skew morphism of a finite group A, let π be the power function of φ, and let k be the order of φ. If φ is smooth, then the following hold:*

(a) *φ is kernel-preserving,*
(b) *$\pi\colon A \to \mathbb{Z}_k$ is a group homomorphism from A to the multiplicative group $\mathbb{Z}_k^*$ with $\operatorname{Ker}\pi = \operatorname{Ker}\varphi$,*
(c) *for any φ-invariant normal subgroup N of A, the induced skew morphism $\bar{\varphi}$ on A/N is also smooth, in particular, if $N = \operatorname{Ker}\varphi$ then $\bar{\varphi}$ is the identity permutation,*
(d) *for any positive integer k, $\mu = \varphi^k$ is a smooth skew morphism,*
(e) *for any automorphism γ of A, $\psi = \gamma^{-1}\varphi\gamma$ is a smooth skew morphism of A.*

In what follows we turn to skew morphisms of cyclic groups.

Definition 9. Let $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of skew morphisms of the cyclic additive groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, and let π and $\widetilde{\pi}$ be the power functions of φ and $\widetilde{\varphi}$, respectively. The pair $(\varphi, \widetilde{\varphi})$ is *reciprocal* if they satisfy the following conditions:

(a) $|\varphi|$ divides m and $|\widetilde{\varphi}|$ divides n,
(b) $\pi(x) \equiv \widetilde{\varphi}^x(1) \pmod{|\varphi|}$ and $\widetilde{\pi}(y) \equiv \varphi^y(1) \pmod{|\widetilde{\varphi}|}$ for all $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$.

The concept of reciprocal pair of skew morphisms was first introduced by Feng et al as an alternative approach to classify regular dessins with complete bipartite underlying graphs [**8**]. Note that the above definition is different from but equivalent to the original one, see [**8**, Corollary 7]. It was proved that the isomorphism classes of regular dessins with complete bipartite underlying graphs $K_{m,n}$ are in one-to-one correspondence with the reciprocal pairs of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ [**8**, Theorem 5]. It follows that, to classify complete regular dessins, it suffices to determine reciprocal pairs of skew morphisms of cyclic groups.

Example 1. For every pair of positive integers m and n, the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ admit at least one reciprocal pair of skew morphisms, that is, the trivial pair $(\mathrm{id}_n, \mathrm{id}_m)$, where id_k denotes the identity automorphism of $\mathbb{Z}_k$, $k = m, n$. In particular, this is the only reciprocal pair of skew morphisms of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ if and only if $\gcd(m, \phi(n)) = \gcd(n, \phi(m)) = 1$ [**8**]. Note that the corresponding complete regular dessin is a regular dessin with underlying graph $K_{m,n}$ and automorphism group $\mathbb{Z}_m \times \mathbb{Z}_n$.

Reciprocal pairs of skew morphisms have the following important properties.

Lemma 10 ([**8**, Corollary 6])**.** *Let $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$, and let π and $\widetilde{\pi}$ be the associated power functions, respectively. Then*

$$\varphi(x) \equiv \sum_{i=1}^{x} \widetilde{\pi}(\widetilde{\varphi}^{i-1}(1)) \pmod{|\widetilde{\varphi}|} \qquad \textit{and} \qquad \widetilde{\varphi}(y) \equiv \sum_{i=1}^{y} \pi(\varphi^{i-1}(1)) \pmod{|\varphi|}.$$

Lemma 11. *Let $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. Then $|\mathbb{Z}_m\colon \operatorname{Ker}\widetilde{\varphi}|$ divides $|\varphi|$ and $|\mathbb{Z}_n\colon \operatorname{Ker}\varphi|$ divides $|\widetilde{\varphi}|$.*

Proof. Set $k = |\mathbb{Z}_m\colon \operatorname{Ker}\widetilde{\varphi}|$ and $\ell = |\mathbb{Z}_n\colon \operatorname{Ker}\varphi|$, then $\operatorname{Ker}\widetilde{\varphi} = \langle k\rangle$ and $\operatorname{Ker}\varphi = \langle \ell\rangle$. Since φ and $\widetilde{\varphi}$ form reciprocal pair, by Definition 9(b), we have

$$\pi(|\widetilde{\varphi}|) = \widetilde{\varphi}^{|\widetilde{\varphi}|}(1) \equiv 1 \pmod{|\varphi|},$$

and

$$\widetilde{\pi}(|\varphi|) = \varphi^{|\varphi|}(1) \equiv 1 \pmod{|\widetilde{\varphi}|},$$

so $|\widetilde{\varphi}| \in \operatorname{Ker}\varphi$ and $|\varphi| \in \operatorname{Ker}\widetilde{\varphi}$. Hence, k divides $|\widetilde{\varphi}|$ and ℓ divides $|\varphi|$, as required. □

Lemma 12. *Let $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ be a reciprocal pair of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. If one of the skew morphisms is an automorphism, then the other is smooth. In particular, if one of the skew morphisms is the identity permutation, then the other is an automorphism.*

Proof. Without loss of generality we may suppose that $\widetilde{\varphi}$ is an automorphism of $\mathbb{Z}_m$. Then $\widetilde{\pi}(y) \equiv 1 \pmod{|\widetilde{\varphi}|}$ for any $y \in \mathbb{Z}_m$. By Lemma 10, $\varphi(x) \equiv \sum_{i=1}^{x} \widetilde{\pi}(\widetilde{\varphi}^{i-1}(1)) \equiv x \pmod{|\widetilde{\varphi}|}$ for any $x \in \mathbb{Z}_n$, and so $\varphi(x) - x$ is a multiple of $|\widetilde{\varphi}|$. By Lemma 11, $|\widetilde{\varphi}|$ is a multiple of $|\mathbb{Z}_n \colon \operatorname{Ker}\varphi|$. Thus $\varphi(x) - x \in \operatorname{Ker}\varphi$, and consequently, $\pi(\varphi(x)) = \pi(x)$. Therefore, by Lemma 7, φ is smooth. In particular, if $\widetilde{\varphi} = \mathrm{id}$, then by Definition 9, $\pi(x) \equiv \widetilde{\varphi}^x(1) \equiv 1 \pmod{|\varphi|}$ for all $x \in \mathbb{Z}_n$, so φ is an automorphism of $\mathbb{Z}_n$. □

3. Main results

In this section, we determine all reciprocal pairs of skew morphisms, provided that one of the skew morphisms is an automorphism. We first prove a technical result as follows.

Lemma 13. *Let n, k, r, s, t be positive integers satisfying the following conditions:*

(1) *k divides n, and $r, s \in \mathbb{Z}_{n/k}$,*

(2) *t has multiplicative order k in $\mathbb{Z}_m$, where m is the smallest positive integer such that $r \sum_{i=1}^{m} s^{i-1} \equiv 0 \pmod{n/k}$,*

(3) *$s^{t-1} \equiv 1 \pmod{n/k}$ and $s - 1 \equiv r \sum_{i=1}^{k} \sum_{j=1}^{t^{i-1}} s^{j-1} \pmod{n/k}$.*

Then the function $\tau(s,t) = \sum_{i=1}^{t} s^{i-1}$ possesses the following properties:

(a) *$(s-1)\tau(s,t) \equiv s - 1 \pmod{n/k}$,*

(b) *for any positive integer i, $\tau(s, t^i) \equiv \tau(s,t)^i \pmod{n/k}$,*

(c) *if k divides i, then $r\tau(s,t)^i \equiv r \pmod{n/k}$,*

(d) *for any positive integers ℓ and u,*

$$r \sum_{j=1}^{k\ell+u} \tau(s,t)^{j-1} \equiv \ell(s-1) + r \sum_{j=1}^{u} \tau(s,t)^{j-1} \pmod{n/k}.$$

Proof. Since $(s-1)\tau(s,t) = s\tau(s,t) - \tau(s,t) = s^t - 1$, by (3), we have $(s-1)\tau(s,t) \equiv s - 1 \pmod{n/k}$. Moreover, for any positive integer i, we have

$$\begin{aligned}
\tau(s, t^{i+1}) &= \sum_{j=1}^{t} s^{j-1} + s^t \sum_{j=1}^{t} s^{j-1} + \cdots + s^{t^{i+1}-t} \sum_{j=1}^{t} s^{j-1} \\
&= (1 + s^t + \cdots + s^{t^{i+1}-t}) \sum_{j=1}^{t} s^{j-1} \\
&\equiv (1 + s + s^2 + \cdots + s^{t^i-1}) \sum_{j=1}^{t} s^{j-1} \\
&\equiv \tau(s, t^i)\tau(s,t) \pmod{n/k}.
\end{aligned}$$

Using induction we obtain $\tau(s,t^i) \equiv \tau(s,t)^i \pmod{n/k}$. In particular, if k divides i, then by (2) we have $t^i \equiv 1 \pmod m$, so

$$r\tau(s,t)^i \equiv r\tau(s,t^i) \equiv r\tau(s,1) \equiv r \pmod{n/k}.$$

Finally, to prove (d), let us denote $\tau := \tau(s,t)$ for brevity. Then by (a) and (c) we have

$$\begin{aligned} r\sum_{j=1}^{k\ell+u}\tau^{j-1} &\equiv r\Big(\sum_{j=1}^{k}\tau^{j-1} + \tau^k\sum_{j=1}^{k}\tau^{j-1} + \cdots + \tau^{k(\ell-1)}\sum_{j=1}^{k}\tau^{j-1}\Big) + r\tau^{k\ell}\sum_{j=1}^{u}\tau^{j-1} \\ &\overset{(c)}{\equiv} (s-1)(1+\tau^k+\cdots\tau^{k(\ell-1)}) + r\sum_{j=1}^{u}\tau^{j-1} \\ &\overset{(a)}{\equiv} \ell(s-1) + r\sum_{j=1}^{u}\tau^{j-1} \pmod{n/k}, \end{aligned}$$

as required. □

Theorem 14. *Every reciprocal pair of skew morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ such that $\widetilde{\varphi}$ is an automorphism of $\mathbb{Z}_m$ is given by the formulae*

$$(1)\qquad \varphi(x) \equiv x + rk\sum_{i=1}^{x}\Big(\sum_{j=1}^{t}s^{j-1}\Big)^{i-1} \pmod n \quad \textit{and} \quad \widetilde{\varphi}(y) \equiv ty \pmod m$$

with the associated power functions determined by the formulae

$$\pi(x) \equiv t^x \pmod{|\varphi|} \qquad \textit{and} \qquad \widetilde{\pi}(y) \equiv 1 \pmod{|\widetilde{\varphi}|},$$

where k,r,s,t are positive integers satisfying the following conditions:

(a) *k is a positive divisor of n, $r \in \mathbb{Z}_{n/k}$, $s \in \mathbb{Z}^*_{n/k}$ and $t \in \mathbb{Z}^*_m$,*

(b) *k divides the multiplicative order of t in $\mathbb{Z}_m$ and the latter divides* $\gcd(rk,n)$,

(c) *if m_1 is the smallest positive integer such that $r\sum_{i=1}^{m_1}s^{i-1} \equiv 0 \pmod{n/k}$, then m_1 divides m and the multiplicative order of t in $\mathbb{Z}_{m_1}$ is equal to k,*

(d) $s-1 \equiv r\sum_{i=1}^{k}\big(\sum_{j=1}^{t}s^{j-1}\big)^{i-1} \pmod{n/k}$,

(e) $s^{t-1} \equiv 1 \pmod{n/k}$.

Conversely, the pair $(\varphi,\widetilde{\varphi})$ defined by the formulae (1) *is a reciprocal pair of skew morphisms of $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $\widetilde{\varphi}$ is an automorphism, provided that the numerical conditions (a)–(e) are satisfied.*

Proof. First assume that $(\varphi,\widetilde{\varphi})$ is a reciprocal pair of skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$ and $\widetilde{\varphi}$ is an automorphism, then $\widetilde{\varphi}\colon y \mapsto ty$ for some $t \in \mathbb{Z}^*_m$. By Lemma 12, the other skew morphism $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is smooth. Set $k = |\mathbb{Z}_n\colon \operatorname{Ker}\varphi|$, then $\operatorname{Ker}\varphi = \langle k\rangle$. By Lemma 8, the induced skew morphism $\bar{\varphi}$

of $\mathbb{Z}_m/\operatorname{Ker}\varphi$ is the identity permutation and the restriction of φ to $\operatorname{Ker}\varphi$ is an automorphism of $\operatorname{Ker}\varphi$. Thus

$$\varphi(1) \equiv 1 + rk \pmod n \quad \text{and} \quad \varphi(k) \equiv sk \pmod n$$

for some $r \in \mathbb{Z}_{n/k}$ and $s \in \mathbb{Z}^*_{n/k}$. By the reciprocality (see Definition 9), we have

$$\pi(x) \equiv \widetilde{\varphi}^x(1) \equiv t^x \pmod{|\varphi|}, \qquad x \in \mathbb{Z}_n.$$

From the identity $\varphi(1) \equiv 1 + rk \pmod n$ we deduce that

$$\varphi^2(1) \equiv \varphi(1+rk) \equiv \varphi(rk) + \varphi(1) \equiv 1 + rk + rks = 1 + rk(1+s) \pmod n.$$

Using induction $\varphi^\ell(1) \equiv 1 + rk\tau(s,\ell) \pmod n$, where $\tau(s,\ell) = \sum\limits_{j=1}^{\ell} s^{j-1}$. Moreover,

$$\varphi(2) \equiv \varphi(1+1) \equiv \varphi(1) + \varphi^{\pi(1)}(1) \equiv \varphi(1) + \varphi^t(1) \equiv 2 + rk(1 + \tau(s,t)) \pmod n.$$

Using induction again we obtain

$$\varphi(x) \equiv x + rk \sum_{i=1}^{x} \tau(s, t^{i-1}) \pmod n, \qquad x \in \mathbb{Z}_n. \tag{2}$$

Combining the above identities we have $sk \equiv \varphi(k) \equiv k + rk\sum_{i=1}^{k} \tau(s,t^{i-1})$ $\pmod n$, which is reduced to

$$s - 1 \equiv r \sum_{i=1}^{k} \tau(s, t^{i-1}) \pmod{n/k}. \tag{3}$$

Furthermore, since $\varphi(k) + \varphi(1) \equiv \varphi(k+1) \equiv \varphi(1+k) \equiv \varphi(1) + \varphi^t(k) \pmod n$, we obtain $\varphi(k) \equiv \varphi^t(k) \pmod n$, which implies that

$$s^{t-1} \equiv 1 \pmod{n/k}.$$

Now by Lemma 2, the order $m_1 = |\varphi|$ is equal to the length of the orbit O_1 of φ containing 1. Thus, m_1 is equal to the smallest positive integer such that $\varphi^{m_1}(1) \equiv 1 \pmod n$, or equivalently, $r\tau(s, m_1) \equiv 0 \pmod{n/k}$. The reciprocality implies that m_1 divides m and $1 = \pi(k) = \widetilde{\varphi}^k(1) = t^k \pmod{m_1}$. It follows from the minimality of k that the multiplicative order of t in $\mathbb{Z}_{m_1}$ is precisely k. By Lemma 13(b), the formula (2) is reduced to the stated form in (1) and the congruence (3) is reduced to the stated condition (d). On the other hand, the order $n_1 = |\widetilde{\varphi}|$ is equal to the multiplicative order of t in $\mathbb{Z}_m$. By Lemma 11, $k = |\mathbb{Z}_n \colon \operatorname{Ker}\varphi|$ divides n_1. By the reciprocality, n_1 divides n and $1 \equiv \widetilde{\pi}(1) = \varphi(1) \equiv 1 + rk \pmod{n_1}$, so n_1 also divides rk.

Conversely, we verify that the pair $(\varphi, \widetilde{\varphi})$ given by (1) is a reciprocal pair of skew morphisms, provided that the numerical conditions (a)–(e) are satisfied.

First, using properties of the function τ proved in Lemma 13 we derive from (1) that

$$\varphi^i(x) = x + rk\tau(s,i) \sum_{j=1}^{x} \tau(s,t)^{j-1}. \tag{4}$$

Second, it is evident that $\widetilde{\varphi}$ is an automorphism of $\mathbb{Z}_m$. By (1) we have

$$\varphi(k) \equiv k + rk \sum_{i=1}^{k} \tau(s,t)^{i-1} \overset{\text{(d)}}{\equiv} sk \pmod{n}.$$

Since $\gcd(s, n/k) = 1$, the restriction of φ to $\langle k \rangle$ is an automorphism of $\langle k \rangle$. Moreover, if $\varphi(x) \equiv \varphi(y) \pmod{n}$ for $x \leq y$, then

$$x + rk \sum_{i=1}^{x} \tau(s,t)^{i-1} \equiv y + rk \sum_{i=1}^{y} \tau(s,t)^{i-1} \pmod{n},$$

or equivalently,

$$0 \equiv y - x + rk \sum_{i=x+1}^{y} \tau(s,t)^{i-1} \equiv y - x + rk\tau(s,t^x) \sum_{i=1}^{y-x} \tau(s,t)^{i-1} \pmod{n}.$$

Thus $x - y \equiv 0 \pmod{k}$, and so $x - y = ku$ for some integer u. It follows that

$$0 \equiv y - x + rk\tau(s,t^x) \sum_{i=1}^{y-x} \tau(s,t)^{i-1} = \varphi^{t^x}(y-x) = \varphi^{t^x}(uk) = s^{t^x}uk \pmod{n},$$

which implies $u \equiv 0 \pmod{n/k}$, and so $x \equiv y \pmod{n}$. Therefore, φ is a bijection on $\mathbb{Z}_n$. By Lemma 13(d), we have $\varphi(0) = 0$. Now for any $x, y \in \mathbb{Z}_n$, by (4), we have

$$\begin{aligned}
\varphi(x+y) &\equiv x + y + rk \sum_{i=1}^{x+y} \tau(s,t)^{i-1} \\
&\equiv (x + rk \sum_{i=1}^{x} \tau(s,t)^{i-1}) + (y + rk \sum_{i=x+1}^{x+y} \tau(s,t)^{i-1}) \\
&\equiv (x + rk \sum_{i=1}^{x} \tau(s,t)^{i-1}) + (y + rk\tau(s,t^x) \sum_{i=1}^{y} \tau(s,t)^{i-1}) \\
&\equiv \varphi(x) + \varphi^{\pi(x)}(y) \pmod{n}.
\end{aligned}$$

Therefore, φ is a skew morphism of $\mathbb{Z}_n$ with kernel $\operatorname{Ker}\varphi = \langle k \rangle$.

Finally, by (b) and (c), we have $|\varphi| = m_1$ and $|\widetilde{\varphi}| = n_1$, and they divide m and n, respectively. On the other hand,

$$\pi(x) = t^x = \widetilde{\varphi}^x(1) \pmod{|\varphi|} \quad \text{and} \quad \varphi^y(1) \equiv 1 + rk\tau(y) \equiv 1 \equiv \widetilde{\pi}(y) \pmod{|\widetilde{\varphi}|}.$$

Consequently, the pair $(\varphi, \widetilde{\varphi})$ of skew morphisms fulfills all conditions in Definition 9, and therefore it is reciprocal, as required. □

Remark 1. From the proof of Theorem 14 we see that the restriction of φ to $\operatorname{Ker}\varphi$ is the identity automorphism of $\operatorname{Ker}\varphi$ if and only if $s = 1$. In this case we have $\tau(s,t) = t$, and the reciprocal pair of skew morphisms φ and $\widetilde{\varphi}$ is given by

$$\varphi(x) = x + rk \sum_{i=1}^{x} t^{i-1} \quad \text{and} \quad \widetilde{\varphi}(y) = ty,$$

where the numerical conditions on the parameters k, r and t are reduced to the following:

(a) k is a divisor of n, $r \in \mathbb{Z}_{n/k}$ and $t \in \mathbb{Z}_m^*$,

(b) k divides the multiplicative order of t in $\mathbb{Z}_m$ and the latter divides $\gcd(rk, n)$,

(c) $m_1 = n/\gcd(n, rk)$ divides m and the multiplicative order of t in $\mathbb{Z}_{m_1}$ is k,

(d) $r \sum_{i=1}^{k} t^{i-1} \equiv 0 \pmod{n/k}$.

Corollary 15. *Let $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ be a pair of reciprocal skew morphisms of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_m$. If both φ and $\widetilde{\varphi}$ are automorphisms, then*

$$\varphi(x) = sx \quad \text{and} \quad \widetilde{\varphi}(y) = ty$$

for some integers $s \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_m^*$ such that

(a) *if m_1 is the multiplicative order of s in $\mathbb{Z}_n$, then m_1 divides m and $t \equiv 1 \pmod{m_1}$,*

(b) *if n_1 is the multiplicative order of t in $\mathbb{Z}_m$, then n_1 divides n and $s \equiv 1 \pmod{n_1}$.*

Proof. Note that the skew morphism φ of $\mathbb{Z}_n$ is an automorphism if and only if $\operatorname{Ker}\varphi = \mathbb{Z}_n$. Thus to prove the corollary we simply put $k = 1$ in Theorem 14, and the simplified conditions are obtained by reduction. We leave the verification to the reader. In what follows we provide a simpler proof which is independent of Theorem 14.

By hypothesis, both φ and $\widetilde{\varphi}$ are automorphisms, so there exist integers $s \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_m^*$ such that $\varphi(x) = sx$ and $\widetilde{\varphi}(y) = ty$ where $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_m$. Let m_1 and n_1 denote the multiplicative orders of s and t in $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively, then $|\varphi| = m_1$ and $|\widetilde{\varphi}| = n_1$. Since $(\varphi, \widetilde{\varphi})$ is reciprocal, by Definition 9(a), m_1 divides m and n_1 divides n. By Definition 9(b) we have $1 \equiv \pi(1) \equiv \widetilde{\varphi}(1) \equiv t \pmod{m_1}$ and $1 \equiv \widetilde{\pi}(1) \equiv \varphi(1) \equiv s \pmod{n_1}$, as required. □

Remark 2. Kovács and Nedela have shown that every skew morphism of $\mathbb{Z}_m$ is an automorphism if and only if $m = 4$ or $\gcd(m, \phi(m)) = 1$ [**15**, Theorem 6.3]. Thus, if $m = 4$ or $\gcd(m, \phi(m)) = 1$, using Theorem 14 we obtain all pairs of reciprocal skew morphisms $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ and $\widetilde{\varphi}\colon \mathbb{Z}_m \to \mathbb{Z}_m$ for any positive integer n.

Remark 3. Depending on the correspondence between complete regular dessins and bicyclic groups with two distinguished generators, Hu, Nedela and Wang have recently classified complete regular dessins with underlying graphs $K_{m,n}$, where both m and n are powers of an odd prime p [**9**]. For example, in the case where $m = 9$ and $n = 27$, up to isomorphism there are precisely 27 complete regular dessins with underlying graph $K_{m,n}$. Their automorphism groups split into the following four families:

(1) 1 regular dessin with automorphism group

$$G = \langle a, b \mid a^9 = b^{27} = [a, b] = 1 \rangle \cong \mathbb{Z}_9 \times \mathbb{Z}_{27}.$$

(2) 2 regular dessins with automorphism groups

$$G = \langle a, b \mid a^9 = b^{27} = 1, b^a = b^{10} \rangle \cong \mathbb{Z}_{27} \rtimes_{10} \mathbb{Z}_9.$$

(3) 18 regular dessins with automorphism groups

$$G = \langle a, b \mid a^9 = b^{27} = 1, b^a = b^4 \rangle \cong \mathbb{Z}_{27} \rtimes_4 \mathbb{Z}_9.$$

(4) 6 regular dessins with automorphism groups

$$G = \langle a, b \mid a^9 = b^{27} = 1, a^b = a^4 \rangle \cong \mathbb{Z}_9 \rtimes_4 \mathbb{Z}_{27}.$$

On the other hand, for $n = 27$ and $m = 9$, Theorem 14 can be used to determine the reciprocal pair of skew morphisms $\varphi\colon \mathbb{Z}_{27} \to \mathbb{Z}_{27}$ and $\widetilde{\varphi}\colon \mathbb{Z}_9 \to \mathbb{Z}_9$ of the cyclic groups $\mathbb{Z}_{27}$ and $\mathbb{Z}_9$. They split into two families

(a) 15 pairs of automorphisms of the form

$$\varphi(x) \equiv sx \pmod{27} \quad \text{and} \quad \widetilde{\varphi}(y) \equiv ty \pmod 9$$

where $(s,t) = (1,1), (19,1), (4,1), (22,1), (7,1), (25,1), (10,1), (13,1), (16,1), (1,4), (1,7), (19,4), (10,4), (19,7), (10,7)$.

(b) 12 pairs of proper (smooth) skew morphisms and automorphisms of the form

$$\varphi(x) \equiv x + rk \sum_{i=1}^{x} \Big(\sum_{j=1}^{t} s^{j-1} \Big)^{i-1} \pmod{27} \qquad \text{and} \qquad \widetilde{\varphi}(y) \equiv ty \pmod 9$$

where $(k,r,s,t) = (3,1,4,4), (3,1,4,7), (3,2,7,4), (3,2,7,7), (3,4,4,4), (3,4,4,7), (3,5,7,4), (3,5,7,7), (3,7,4,4), (3,7,4,7), (3,8,7,4), (3,8,7,7)$.

Thus, the total number of reciprocal pairs of skew morphisms of the cyclic groups $\mathbb{Z}_{27}$ and $\mathbb{Z}_9$ is equal to $15 + 12 = 27$, as expected.

4. Smooth skew morphisms of cyclic groups revisited

Under the name of coset-preserving skew morphisms, smooth skew morphisms of cyclic groups were first classified by Bachratý and Jajcay in [**2**]. In this section using the ideas developed in the proof of Theorem 14 we give a new formulation of their result and present a much simpler proof.

Theorem 16. *For $n > 1$, the proper smooth skew morphisms φ of the cyclic group $\mathbb{Z}_n$ are in one-to-one correspondence with the quadruples (k,r,s,t) of positive integers satisfying the following numerical conditions*

(a) *$k > 1$ is a proper divisor of n, $r \in \mathbb{Z}_{n/k}$ and $s \in \mathbb{Z}^*_{n/k}$,*

(b) *the number t has multiplicative order k in $\mathbb{Z}_m$, where m is the smallest positive integer m such that $r \sum_{i=1}^{m} s^{i-1} \equiv 0 \pmod{n/k}$,*

(c) *$s - 1 \equiv r[(\sum_{i=1}^{t} s^{i-1})^k - 1)]/(\sum_{i=1}^{t} s^{i-1} - 1) \pmod{n/k}$,*

(d) *$s^{t-1} \equiv 1 \pmod{n/k}$.*

Moreover, the skew morphism φ *of* $\mathbb{Z}_n$ *and the associated power function* π *are given by*

$$\varphi(x) \equiv x + rk\frac{\tau(s,t)^x - 1}{\tau(s,t) - 1} \pmod{n} \quad \text{and} \quad \pi(x) \equiv t^x \pmod{m}. \tag{5}$$

Proof. First assume that φ is a proper smooth skew morphism of the cyclic additive group $\mathbb{Z}_n$, $n > 1$. Then $\operatorname{Ker}\varphi = \langle k\rangle$ where $k = |\mathbb{Z}_n\colon \operatorname{Ker}\varphi|$. In particular, k is a divisor of n. Since φ is a proper skew morphism, $\operatorname{Ker}\varphi < \mathbb{Z}_n$, and hence $k > 1$. By Lemma 8, the induced skew morphism $\bar{\varphi}$ on $\mathbb{Z}_n/\operatorname{Ker}\varphi$ is the identity permutation, the restriction of φ to $\operatorname{Ker}\varphi$ is an automorphism of $\operatorname{Ker}\varphi$ and the power function π of φ is a group homomorphism of $\mathbb{Z}_n$ into $\mathbb{Z}^*_{|\varphi|}$. Thus

$$\varphi(1) \equiv 1 + rk \pmod{n}, \quad \varphi(k) \equiv sk \pmod{n} \quad \text{and} \quad \pi(x) \equiv t^x \pmod{|\varphi|},$$

where $r \in \mathbb{Z}_{n/k}$, $s \in \mathbb{Z}^*_{n/k}$ and $t \in \mathbb{Z}^*_{|\varphi|}$. As in proof of Theorem 14, from the above equations it is easy to derive the formulae (5). It follows that

$$sk \equiv \varphi(k) = k + rk\frac{\tau(s,t)^k - 1}{\tau(s,t) - 1} \pmod{n},$$

and so

$$s - 1 \equiv \frac{r(\tau(s,t)^k - 1)}{\tau(s,t) - 1} \pmod{n/k}.$$

where $\tau(s,t) = \sum_{i=1}^{t} s^{i-1}$. Furthermore,

$$sk + (1 + rk) \equiv \varphi(k+1) \equiv \varphi(1+k) \equiv \varphi(1) + \varphi^t(k) = (1 + rk) + s^t k \pmod{n},$$

which implies that $s^{t-1} \equiv 1 \pmod{n/k}$.

By Lemma 2, the order $m = |\varphi|$ is equal to the length of the orbit O_1 of φ containing 1, so m is equal to the smallest positive integer such that $\varphi^m(1) \equiv 1 \pmod{n}$. By (5) we obtain $1 + rk\tau(s,m) \equiv 1 \pmod{n}$, or equivalently, $r\tau(s,m) \equiv 0 \pmod{n/k}$. Since $1 = \pi(k) \equiv t^k \pmod{m}$, the minimality of k implies that the multiplicative order of t is precisely k.

Conversely, we verify that φ given by (5) is indeed a proper smooth skew morphism of $\mathbb{Z}_n$ with $\operatorname{Ker}\varphi = \langle k\rangle$, provided that the numerical conditions (a)–(d) are fulfilled.

First, using properties of the function τ proved in Lemma 13 we derive from (5) that

$$\varphi^\ell(x) \equiv x + rk\tau(s,\ell)\frac{\tau(s,t)^x - 1}{\tau(s,t) - 1} \pmod{n}. \tag{6}$$

Now suppose that $\varphi(x) \equiv \varphi(y) \pmod{n}$ where $x \le y$, then by (5) we have

$$x + rk\frac{\tau(s,t)^x - 1}{\tau(s,t) - 1} \equiv y + rk\frac{\tau(t)^y - 1}{\tau(s,t) - 1} \pmod{n},$$

or equivalently,

$$0 \equiv (y - x) + rk\tau(s,t)^x\frac{\tau(s,t)^{y-x} - 1}{\tau(s,t) - 1} \pmod{n}.$$

Thus $x \equiv y \pmod{k}$. Set $y - x = ku$. The above congruence is reduced to

$$0 \equiv \varphi^{t^x}(y-x) \equiv \varphi^{t^x}(ku) \equiv s^{t^x} ku \pmod{n},$$

so $u \equiv 0 \pmod{n/k}$ and hence $x \equiv y \pmod{n}$. Therefore φ is a bijection on $\mathbb{Z}_n$. Clearly, $\varphi(0) \equiv 0 \pmod{n}$.

Next, for any $x, y \in \mathbb{Z}_n$, by (6) we have

$$\begin{aligned}\varphi(x+y) &= x + y + rk\frac{\tau(s,t)^{x+y} - 1}{\tau(s,t) - 1} \\ &= x + y + rk\frac{(\tau(s,t)^x - 1) + (\tau(s,t)^x\tau(s,t)^y - \tau(s,t)^x)}{\tau(s,t) - 1} \\ &= \Big(x + rk\frac{\tau(s,t)^x - 1}{\tau(s,t) - 1}\Big) + \Big(y + rk\tau(s,t^x)\frac{\tau(s,t)^y - 1}{\tau(s,t) - 1}\Big) \\ &= \varphi(x) + \varphi^{t^x}(y) = \varphi(x) + \varphi^{\pi(x)}(y).\end{aligned}$$

Therefore φ is a skew morphism of $\mathbb{Z}_n$. Clearly, $\operatorname{Ker}\varphi = \langle k \rangle$. Since $k > 1$ is a proper divisor of n, $\operatorname{Ker}\varphi < \mathbb{Z}_n$, so φ is a proper skew morphism. Since

$$\pi(\varphi(x)) = \pi\big(x + rk\frac{\tau(s,t)^x - 1}{\tau(s,t) - 1}\big) \equiv \pi(x) \pmod{m}$$

for all $x \in \mathbb{Z}_n$, by Lemma 7, φ is smooth. From the proof it is easily seen that two such skew morphisms are identical if and only if the corresponding parameters (k, r, s, t) are identical, as required. □

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